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On the decay of isotropic turbulence

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We investigate the decay of freely evolving isotropic turbulence. There are two canonical cases: $E(k \to 0) \sim Lk^2$ and $E(k \to 0) \sim Ik^4$, L and I being the Saffman and Loitsyansky integrals respectively. We focus on the second of these. Numerical simulations are performed in a periodic domain whose dimensions, l_{box}, are much larger than the integral scale of the turbulence, l. We find that, provided that $l_{box} \gg l$ and $Re \gg 1$, the turbulence evolves to a state in which I is approximately constant and Kolmogorov's classical decay law, $u^2 \sim t^{-10/7}$, holds true. The approximate conservation of I in fully developed turbulence implies that the long-range interactions between remote eddies, as measured by the triple correlations, are very weak. This finding seems to be at odds with the non-local nature of the Biot-Savart law.

1. Introduction

It has been known since the classical work of Birkhoff, Batchelor and Saffman (Batchelor 1953; Birkhoff 1954; Batchelor & Proudman 1956; Saffman 1967) that there are two canonical problems in decaying, isotopic turbulence. On the one hand we might have a Birkhoff–Saffman spectrum, $E(k \to 0) = Lk^2/4\pi^2$, where $L = \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, d\boldsymbol{r}$ is known as Saffman's integral. (Here $\boldsymbol{u}$ and $\boldsymbol{u}'$ are the velocities at two points separated by the displacement vector $\boldsymbol{r}$.) Saffman showed that L is an invariant of this motion, reflecting the global conservation of linear momentum, and that, as a consequence, the kinetic energy of the turbulence, $\frac{1}{2}\langle \boldsymbol{u}^2 \rangle = \frac{3}{2}u^2$, decays as $u^2 \sim t^{-6/5}$. On the other hand, in those cases where $L = 0$ we usually have $E(k \to 0) = Ik^4/24\pi^2$, where $I = -\int \boldsymbol{r}^2 \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \, d\boldsymbol{r}$ is known as Loitsyansky's integral. Whether turbulence is of the Batchelor ($E \sim k^4$) or Saffman ($E \sim k^2$) type depends on the initial condition. If the turbulence is created by a mechanism which imparts a significant amount of linear impulse to the fluid, then L is non-zero and $E(k) \sim k^2$. Conversely, if the turbulence is created with little linear impulse then the initial value of L is zero and, since L is an invariant, it stays zero. Either form of turbulence may be created in computer simulations, but it is an open question as to which is more prevalent in nature. Note that the fact that $I = 0$ does not necessarily imply that $E \sim k^4$, since $E \sim k^n$, $2 < n < 4$, is also a theoretical possibility (see, for example, Birkhoff 1954 or Eyink & Thomson 2000). However, there are reasons for believing that $n = 2$ and $n = 4$ have a special status in real turbulence. For example, for $n = 2$ and $n = 4$ the pre-factors multiplying k^n have simple physical interpretations in terms of the linear and angular momentum of the turbulence, and the invariance of these pre-factors, when applicable, is simply related to the conservation of these physical quantities (see, for example, Davidson 2004). This is not true for values other than 2 and 4. Also, the spectrum tensor is singular at $\boldsymbol{k}=\mathbf{0}$ for all n other than 4. In any event, turbulence which starts as k^2 stays as k^2, and turbulence which starts as k^4 (or steeper) stays as k^4.

In this paper we restrict ourselves to Batchelor ($E\sim k^4$) turbulence. We are interested in the behaviour of Loitsyansky's integral, $I(t)$, and in the associated question of the decay exponent in the energy decay law $u^2\sim t^{-m}$. These issues have been a source of much debate and controversy and so, to place our work in context, we start with a brief description of the competing theories.

1.1. *The classical theories of freely decaying isotropic turbulence*

In a little discussed paper Kolmogorov (1941) predicted that isotropic turbulence should decay as $u^2\sim t^{-10/7}$. He made three assumptions: (i) the energy decays as $\mathrm{d}u^2/\mathrm{d}t=-Au^3/l$, where l is the integral scale and A is independent of time; (ii) the large scales (but not the whole spectrum) evolve in a self-similar manner when r is normalized by the integral scale; and (iii) Loitsyansky's integral is constant. Combining (ii) and (iii) gives $I\sim u^2l^5=\text{constant}$ which, when substituted into the energy equation, yields $u^2(t)\sim t^{-10/7}$, $l\sim t^{2/7}$. These are Kolmogorov's decay laws. Note that, although Kolmogorov assumed self-similarity of the large scales, it is sometimes claimed that such an assumption is not needed. The idea is that a new integral scale, $\hat{l}$, is defined via the expression $I\sim u^2\hat{l}^5$ and then combined with $\mathrm{d}u^2/\mathrm{d}t=-Cu^3/\hat{l}$, where C is a constant, to give the 10/7 decay law. However, the empirical law $\mathrm{d}u^2/\mathrm{d}t=-Au^3/l$, $A=\text{constant}$, has only been verified for l defined in the more conventional way, as the integral of the longitudinal correlation function. If the turbulence is not self-similar then we have no right to replace $\mathrm{d}u^2/\mathrm{d}t=-Au^3/l$, $A=\text{constant}$, by $\mathrm{d}u^2/\mathrm{d}t=-Cu^3/\hat{l}$, $C=\text{constant}$. Thus, one way or another, self-similarity of the large scales is required.

While assumptions (i) and (ii) have been verified experimentally and are rarely questioned, the third has been heavily criticized. The claim that I is an invariant originated with Loitsyansky (1939) who noted that the Kármán–Howarth equation

$$\frac{\partial}{\partial t}[u^2r^4f(r,t)] = u^3\frac{\partial}{\partial r}[r^4K(r)] + 2\nu u^2\frac{\partial}{\partial r}[r^4f'(r)] \tag{1.1}$$

may be integrated to give

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[u^2\int_0^\infty r^4 f(r)\,\mathrm{d}r\right] = [u^3r^4K]_\infty + 2\nu[u^2r^4f'(r)]_\infty, \tag{1.2}$$

where $u^2f(r)=\langle u_x(\boldsymbol{x})\,u_x(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle$, $u^3K(r)=\langle u_x^2(\boldsymbol{x})\,u_x(\boldsymbol{x}+r\hat{\boldsymbol{e}}_x)\rangle$ and the subscript ∞ indicates the asymptotic value at large r. If, as assumed by Loitsyansky and Kolmogorov, remote points are statistically independent, in the sense that f and K decay exponentially fast at large r, then

$$I = -\int \boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r} = 8\pi u^2\int_0^\infty r^4 f\,\mathrm{d}r = \text{constant}. \tag{1.3}$$

So, provided the statistical correlations between remote points in a turbulent flow are sufficiently weak, Kolmogorov's decay laws should hold. However, as we shall see, the strength of these long-range correlations has been hotly disputed. In any event, Loitsyansky's claim was given some physical basis by Landau (Landau & Lifshitz 1959) who noted that the conservation of I could be attributed to the general principle of conservation of angular momentum. In particular, he showed that

$$I = -\int \boldsymbol{r}^2\langle\boldsymbol{u}\cdot\boldsymbol{u}'\rangle\,\mathrm{d}\boldsymbol{r} = \langle\boldsymbol{H}^2\rangle/V = \text{constant} \tag{1.4}$$

where $\boldsymbol{H}$ is the angular momentum in some large volume V, $\boldsymbol{H} = \int (\boldsymbol{x} \times \boldsymbol{u})\,\mathrm{d}V$. However, like Loitsyansky, Landau had to assume that the long-range correlations are negligible. Moreover, he assumed that the turbulence has zero net linear impulse, thus excluding a Saffman spectrum (see, for example, Davidson 2004).

The first doubts over the validity of (1.3), and hence the 10/7 decay law, came with the work of Proudman & Reid (1954) who showed that the quasi-normal (QN) closure model yields

$$\frac{\mathrm{d}}{\mathrm{d}t}[u^3 r^4 K]_\infty = \tfrac{7}{10} 4\pi \int_0^\infty (E^2/k^2)\,\mathrm{d}k$$

in isotropic turbulence. Combining this with (1.2) and (1.3) we have

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8\pi \frac{\mathrm{d}}{\mathrm{d}t}[u^3 r^4 K]_\infty = \tfrac{7}{5}(4\pi)^2 \int_0^\infty (E^2/k^2)\,\mathrm{d}k = \tfrac{14}{5} \int \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle^2\,\mathrm{d}\boldsymbol{r} \tag{1.5}$$

where the final equality on the right is a consequence of Rayleigh's power theorem. Of course, from a dynamical point of view, the QN model is deeply flawed. Nevertheless, random Gaussian modes represent a perfectly legitimate initial condition. The implication is that one can create kinematically admissible velocity fields which induce long-range interactions of the form $K \sim r^{-4}$.

The origin of the discrepancy between Proudman & Reid and Loitsyansky was identified by Batchelor & Proudman (1956). They considered homogeneous, anisotropic turbulence and adopted initial conditions in which the fourth-order cumulants, $[u_i u'_j u''_k u_l]_{\mathrm{cum}}$, as well as the second- and third-order velocity correlations, are exponentially small at large separation. They then looked for the growth of algebraic tails in the second- and third-order velocity correlations immediately after $t = 0$. In short, they ignored long-range correlations as far as the fourth-order cumulants are concerned, but allowed for long-range interactions in the second- and third-order correlations. (Note that this is not the same as the QN approximation, which requires that fourth-order cumulants are zero for all values of $\boldsymbol{r}$, large or small.) Their key observation was that an eddy located at point $\boldsymbol{x}$ sets up a pressure field $p' = p(\boldsymbol{x}')$ which falls off as $r^{-3} = |\boldsymbol{x}' - \boldsymbol{x}|^{-3}$ in the far field. Thus, for example, the pressure fluctuations at $\boldsymbol{x}' = \boldsymbol{x} + r\hat{\boldsymbol{e}}_x$, caused by the eddy at $\boldsymbol{x}$, have an intensity $p'_\infty \sim r^{-3}$. This, in turn, sets up a long-range correlation of the form $\langle u_x^2 p' \rangle_\infty \sim r^{-3}$. Moreover, the triple correlations are governed by an equation of the type

$$\rho \frac{\partial}{\partial t}\langle u_i u_j u'_k \rangle = \rho \nabla \cdot [\langle uuuu' \rangle + \langle uuu'u' \rangle] - \frac{\partial}{\partial r_k}\langle u_i u_j p' \rangle - \left\langle u'_k \left(u_i \frac{\partial p}{\partial x_j} + u_j \frac{\partial p}{\partial x_i} \right) \right\rangle \tag{1.6}$$

and so we conclude that the long-range velocity–pressure correlations induce triple velocity correlations of the form $\langle u_i u_j u'_k \rangle_\infty \sim C_{ijk} r^{-4}$. Although Batchelor and Proudman found no long-range correlations when the symmetries of isotropy are imposed (see p. 400 of Batchelor & Proudman 1956), it is usually assumed that $u^3 K(r)$ does indeed decay as $u^3 K(r) \sim C/r^4$ in isotropic turbulence. This implies that I is time dependent and casts doubt on the 10/7 decay law. (Note that, if $K(r)$ decays as $ar^{-4} + br^{-5} + O(r^{-6})$, and there are no long-range correlations at $t = 0$, the Kármán–Howarth equation tells us that the double velocity and vorticity correlations decay as r^{-6} and r^{-8}, respectively.)

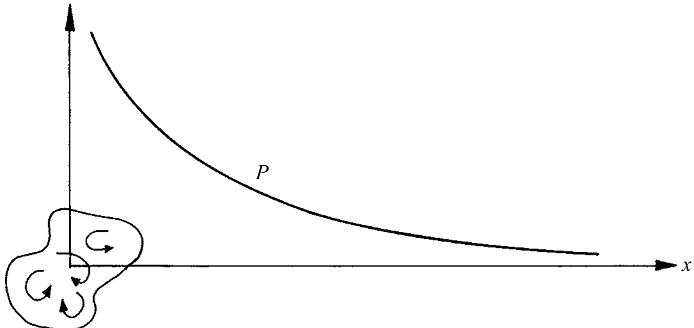

FIGURE 1. The pressure induced at point $\boldsymbol{x}' = r\hat{\boldsymbol{e}}_x$ by eddies in the vicinity of point $\boldsymbol{x} \sim 0$.

1.2. *The results of the closures and simulations*

The suggestion that I should be time-dependent is also seen in certain closure approximations. For example, EDQNM suggests

$$\frac{\mathrm{d}I}{\mathrm{d}t} = 8\pi[u^3 r^4 K]_\infty = \tfrac{7}{5}(4\pi)^2 \int_{k_0}^{\infty} \theta(k)[E^2/k^2]\,\mathrm{d}k \tag{1.7}$$

where k_0 is of the order of l^{-1} and θ (k) is a model-dependent function which has the dimensions of time. A comparison of (1.7) with the quasi-normal estimate (1.5) shows that a time derivative has been removed as a result of Markovianization and replaced by the model parameter θ. In effect, estimate (1.7) amounts to an assertion that the long-range triple correlations do exist, with their strength being set by the magnitude of the parameter θ. The EDQNM closure predicts a slow growth in Loitsyansky's integral, $I \sim t^{0.16}$, with a corresponding energy decay rate of $u^2 \sim t^{-1.38}$. This is somewhat slower than the 10/7 law. These results are not inconsistent with the direct numerical simulations (DNS) of Herring *et al.* (2005) and the large-eddy simulations (LES) of Chasnov (1993) and Ossia & Lesieur (2000). They are, however, inconsistent with the DNS reported here, where we find that $I(t)$ approaches a constant value after an initial transient.

1.3. *The physical interpretation of the Batchelor & Proudman and Proudman & Reid*

Batchelor & Proudman (1956) restricted their claims about long-range correlations to anisotropic turbulence and were unable to find any long-range interactions in the isotropic case. However, their analysis has been extended to isotropic turbulence by Davidson (2000, 2004), and it turns out that all the essential ideas carry over more or less intact. We shall briefly summarize these more recent studies because it clarifies the physical origin of expressions such as (1.5) and Batchelor's estimate $\langle u_x^2 p'\rangle_\infty \sim r^{-3}$. We give only the key results here, leaving the mathematical details to § 2.

Consider a collection of turbulent eddies which lie within a volume V centred on the point $\boldsymbol{x}$, and which form part of a larger cloud of turbulence (figure 1). The far-field pressure induced by these eddies at point $\boldsymbol{x}' = \boldsymbol{x} + r\hat{\boldsymbol{e}}_x$ can be calculated by inverting the pressure equation

$$\nabla^2(p/\rho) = -\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u}) = \tfrac{1}{2}\boldsymbol{\omega}^2 - S_{ij}S_{ij} \tag{1.8}$$

which, using (2.5), yields

$$p'_\infty = \frac{\rho}{4\pi r^3}\int[(2x^2 - y^2 - x^2)\nabla\cdot(\boldsymbol{u}\cdot\nabla\boldsymbol{u})]\,\mathrm{d}\boldsymbol{x} + (\text{surface integrals over } V \text{ of } u_i u_j) + O(r^{-4})$$

or equivalently,

$$p'_\infty = \frac{\rho}{4\pi r^3}\int \left[2u_x^2 - u_y^2 - u_z^2\right]\,\mathrm{d}\boldsymbol{x} + (\text{surface integrals over } V \text{ of } u_i u_j) + O(r^{-4}). \quad (1.9)$$

(Note that, throughout the paper, we take $O(r^{-n})$ to mean bounded asymptotically by a constant times r^{-n}.) Now suppose the volume V has a characteristic radius much greater than l but much smaller than r. It follows that, provided the fourth-order cumulants at large separation can be neglected, so that we can dispense with the surface integrals in (1.9), there exists a pressure–velocity correlation of the form

$$\left\langle u_x^2 p'\right\rangle_\infty = \frac{\rho}{4\pi r^3}\int \left\langle u_x^2[2(u_x^*)^2 - (u_y^*)^2 - (u_z^*)^2]\right\rangle \mathrm{d}\boldsymbol{x}^* + O(r^{-4}). \quad (1.10)$$

This is the origin of Batchelor's result $\langle u_x^2 p'\rangle_\infty \sim r^{-3}$. Note that we expect the large scales to contribute most to the integral on the right of (1.10), suggesting that it is of the order of $u^4 l^3$. It follows that $\langle u_x^2 p'\rangle_\infty \sim u^4(r/l)^{-3}$. We now combine (1.10) with (1.6) and take advantage of isotropy. It turns out that, when fourth-order cumulants at large separation can be neglected, only the second term on the right of (1.6) is important for large r (see § 2), and we obtain

$$\frac{\partial}{\partial t}\left\langle u_x^2 u_x'\right\rangle_\infty = \frac{\partial}{\partial t}[u^3 K]_\infty = \frac{3}{4\pi r^4}\int \langle s s^*\rangle\,\mathrm{d}\boldsymbol{x}^* \quad (1.11)$$

where $s = u_x^2 - u_y^2$. It follows from (1.2) that I is governed by

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8\pi\frac{\mathrm{d}}{\mathrm{d}t}[u^3 r^4 K]_\infty = 6\int \langle s s^*\rangle\,\mathrm{d}\boldsymbol{r}' = 6J, \quad (1.12)$$

where $\boldsymbol{r}' = \boldsymbol{x}^* - \boldsymbol{x}$ (Davidson 2000). The origin of the QN estimate (1.5) is now clear. If we insist that fourth-order cumulants are zero for all $\boldsymbol{r}'$, large and small (and they are definitely not), then $6J = (14/5)\int \langle \boldsymbol{u}\cdot\boldsymbol{u}'\rangle^2\,\mathrm{d}\boldsymbol{r}$ and (1.12) reduces to (1.5). In any event, whether or not we adopt the QN hypothesis, (1.12) predicts that $\mathrm{d}^2 I/\mathrm{d}t^2 > 0$, since J is strictly positive:

$$J = \lim_{V\to\infty}\frac{1}{V}\left[\int s\,\mathrm{d}V\right]^2.$$

Note that (1.11) and (1.12) depend critically on the assumption that fourth-order cumulants decay rapidly for large separation r. (We shall see in § 2 that, in this context, rapidly means faster than r^{-7}.) There is some experimental evidence to suggest that, in fully developed turbulence, the cumulants are indeed small at large separation (Van Atta & Yeh 1970), though it is not possible to tell if they decay faster or slower than r^{-7}. In any event, there are no grounds for believing that (1.11) or (1.12) apply to the transient evolution of non-asymptotic turbulence from random initial conditions, since we do not know how the cumulants behave in such cases.

Essentially the same results can be obtained directly form the Biot-Savart law (Davidson 2004). Again consider our group of eddies confined to a volume V and located near point $\boldsymbol{x}$. If they have negligible linear impulse (we want to avoid a Saffman spectrum) then the dipole field induced by the eddies is zero and the Biot-Savart law gives a far-field velocity at $\boldsymbol{x}' = \boldsymbol{x} + r\hat{\boldsymbol{e}}_x$ of

$$(u_x')_\infty = \frac{3}{4\pi r^4}\int (x^* y^* \omega_z^* - x^* z^* \omega_y^*)\,\mathrm{d}\boldsymbol{x}^* + O(r^{-5}) \quad (1.13)$$

where $\boldsymbol{\omega}$ is the vorticity. (Note that, once again, we have made use of (2.5).) Differentiating with respect to time we find, after a little algebra

$$\frac{\partial u_x'}{\partial t} = \frac{3}{4\pi r^4}\int \left[2u_x^2 - u_y^2 - u_z^2\right]^* \mathrm{d}\boldsymbol{x}^* + (\text{surface integrals of } u_i u_j) + O(r^{-5}). \quad (1.14)$$

It follows that, provided the second-order velocity correlations and fourth-order cumulants at large separation can be neglected, so that we can dispense with the surface integrals in (1.14), there exist long-range triple correlations of the form

$$\frac{\partial}{\partial t}\left\langle u_x^2 u_x' \right\rangle_\infty = \frac{3}{4\pi r^4}\int \left\langle u_x^2\left(2u_x^2 - u_y^2 - u_z^2\right)^*\right\rangle \mathrm{d}\boldsymbol{x}^*. \quad (1.15)$$

We have arrived back at (1.11) and (1.12), the link between the two arguments being that the right of (1.14) is simply the pressure gradient associated with (1.9). Once again we conclude that I is probably time dependent. This, in turn, suggests that, provided the large scales are self-similar, the 10/7 decay law is invalid.

1.4. *Deviations from the 10/7 decay law caused by long-range interactions*

The concluding statement above requires some support. The proof goes as follows. If the energy decays as a power law, $u^2 \sim t^{-m}$, and we shall see that it does at large time, then our energy equation becomes

$$\frac{\mathrm{d}u^2}{\mathrm{d}t} = -A\frac{u^3}{l} = -m\frac{u^2}{t}, \quad (1.16)$$

from which we see that we are at liberty to take the integral scale as $l = ut$. Next, like Kolmogorov, we assume that the large scales are self-similar, which is in line with the experimental evidence for fully developed turbulence (Batchelor 1953). Equation (1.2) then reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}[u^7 t^5] = \lambda u^7 t^4, \qquad \lambda = [\zeta^4 K(\zeta)]_\infty \bigg/ \int \zeta^4 f(\zeta)\,\mathrm{d}\zeta \quad (1.17)$$

where $\zeta = r/l$ and λ is some unknown coefficient. Note that λ is zero if, and only if, I is a constant. From (1.17) we have

$$m = 10/7 - 2\lambda/7 \quad (1.18)$$

which confirms that any time dependence of I invalidates the 10/7 law, at least for fully developed turbulence. Note that m is less than 10/7 when I increases with time, and greater than 10/7 when I is a decreasing function of time. Note also that (1.16), and hence (1.18), assumes $Re \gg 1$, so the rate of loss of energy is independent of viscosity.

1.5. *Uncertainties and questions*

The central observation of Batchelor & Proudman, that long-range correlations are an inevitable consequence of the non-local pressure field, the tentative predictions of the EDQNM closure model, and the results of the numerical simulations, make a convincing case against the 10/7 decay law. However, a more careful analysis shows that the situation is not so clear cut. Let us consider the analysis of Batchelor & Proudman (1956) and its extension by Davidson (2000, 2004). All of its quantitative predictions rest on the assumption that fourth-order cumulants decay rapidly for well-separated points. (Recall that, in §2 we shall see that, in this context, rapidly means faster than r^{-7}.) There is no evidence that this is the case. Thus, for example, equation (1.12) must be regarded with considerable suspicion. And even if we do accept this equation, how are we to determine the integral J without some *ad hoc* closure model? Perhaps the most we can extract from these various studies is the observation that, as a consequence of the Biot-Savart law, there is a likelihood of long-range interactions between distant eddies.

The results of the closures and numerical simulations are also indecisive. In EDQNM there are many *ad hoc* assumptions, not the least of which is the Markovianization procedure itself, which cannot be justified for the large scales. It is possible, therefore, that non-Markovianized closures might perform better in this respect, though we have no evidence of this. In any event we shall see that there is some evidence that the evolving morphology or non-Gaussian statistics of the vorticity field plays an important role in the behaviour of I, and this kind of detailed information is, to some extent, absent in many closure models. The numerical simulations may also be questioned. For example, the DNS have all been carried out in computational domains which are only a few multiples of the integral scale, yet the entire analysis of § 1.1 rests on the assumption that the domain is at least an order of magnitude greater than l. The LES, on the other hand, involves a truncation of the energy spectrum at large k and we cannot rule out the possibility that such a truncation influences the large scales, perhaps through a change in the morphology of the vorticity field.

All of this suggests that we should be cautious about dismissing the Kolmogorov decay law. This opens up the intriguing possibility that the 10/7 law may indeed represent the decay of fully developed turbulence, if not its transition from some prescribed (and unphysical) initial condition.

Perhaps there is a clue in the simulations of Ossia & Lesieur (2000). The initial condition in such simulations consists of random Gaussian modes with no phase coherence, and so the vorticity field is more or less structureless. Fully developed turbulence, on the other hand, has a very intricate vorticity field, consisting of clusters or networks of fine-scale tube-like vortices (Kaneda *et al.* 2004; Kaneda & Ishihara 2006). The LES of Ossia & Lesieur tentatively suggests that the time dependence of I, and by inference the strength of the long-range correlation $[r^4 K]_\infty$, decreases as the vorticity field becomes teased out into fine-scale tubes. But why should these long-range correlations, induced by the Biot-Savart law, progressively weaken as the turbulence matures? This is an issue which, so far, has not been addressed in the literature. However, we might note that analogous behaviour is seen in other physical systems possessing many degrees of freedom. The most famous example is, perhaps, Debye–Huckel screening in plasmas and electrolytes, where the long-range Coulomb force between distant ions shuts down as a result of the clustering of oppositely signed charges (Jackson 1975). Such a clustering reduces the Gibbs free energy of the system, E- TS. (Here E is the electrostatic energy, T the temperature and S the entropy.) Another example occurs in arrays of magnetic dipoles. Here the long-range Lorentz force between distant dipoles is diminished through a pairing or clustering of oppositely signed dipoles. Again, this can be viewed as an energy minimization process, as the magnetic energy associated with the long-range forces is reduced. Such behaviour led Ruelle (1990) to speculate that there should be an analogue of Debye screening in turbulence, in which the two-point vorticity correlation decays exponentially with separation, rather than the power law suggested by the Biot-Savart law. Interestingly, he suggested that turbulence whose vorticity field consists of thin, tube-like structures is most likely to favour such screening.

All of these uncertainties led us to revisit this entire problem. The questions we seek to answer are:

(i) do the fourth-order cumulants decay fast enough for (1.12) to hold?

(ii) does turbulence tend to evolve towards a state in which I is approximately constant?

(iii) if (ii) is true, do we recover Kolmogorov's 10/7 decay law in fully developed turbulence?

We emphasize that these questions are not just relevant to isotropic turbulence. Many other systems, such as homogeneous MHD turbulence, and rotating stratified turbulence, conserve one or more components of angular momentum and hence, if the long-range interactions are weak, possess a Loitsyansky-like invariant of the form of (1.4). (See the discussion in Davidson 1997, 2004.) If there is evidence of the suppression of long-range interactions in isotropic turbulence, there may be reasons to suppose these interactions are also weak in these more complex, fully developed flows.

2. Pressure-induced long-range correlations

As a prelude to presenting the numerical evidence, we explore the nature of the pressure-induced long-range interactions. Our aim is to provide a detailed proof of equations (1.11) and (1.12), which were first suggested in Davidson (2000, 2004), and also to determine the precise conditions under which they hold. In particular, we are interested in the restriction that the fourth-order cumulants must decay rapidly with distance. Two important questions, which have not previously been addressed, are: (i) how rapidly must the cumulants fall for (1.12) to be valid; and (ii) is such a fall-off likely to be observed in practice? We attempt to answer both of these questions here.

We shall show that (1.12) holds only if the fourth-order cumulants decay faster than r^{-7} at large separation, and that it is unlikely that this condition holds in general. Our starting point is the extension of Proudman & Reid (1956) by Davidson (2004). Let us introduce the third-order correlation $S_{ij,k}=\langle u_i(\boldsymbol{x})u_j(\boldsymbol{x})u_k(\boldsymbol{x}')\rangle$. It is readily confirmed that, if viscous forces are neglected,

$$\rho\frac{\partial S_{11,1}}{\partial t}=-\rho\frac{\partial}{\partial x_\ell}\left\langle u_1^2u_1'u_\ell\right\rangle-2\left\langle u_1\frac{\partial p}{\partial x_1}u_1'\right\rangle-\rho\frac{\partial}{\partial x_\ell'}\left\langle u_1^2u_1'u_\ell'\right\rangle-\frac{\partial}{\partial x_1'}\left\langle u_1^2p'\right\rangle \tag{2.1}$$

where $S_{11,1}(r\hat{\boldsymbol{e}}_1)=u^3K(r)=\langle u_x^2u_x'\rangle$. Also, inverting the pressure equation (1.8) yields

$$\left.\begin{aligned}p(\boldsymbol{x})/\rho&=\frac{1}{4\pi}\int\frac{1}{|\boldsymbol{x}^*-\boldsymbol{x}|}\frac{\partial^2}{\partial x_m^*\partial x_n^*}[u_m^*u_n^*]\,\mathrm{d}\boldsymbol{x}^*,\\ p(\boldsymbol{x}')/\rho&=\frac{1}{4\pi}\int\frac{1}{|\boldsymbol{x}^*-\boldsymbol{x}'|}\frac{\partial^2}{\partial x_m^*\partial x_n^*}[u_m^*u_n^*]\,\mathrm{d}\boldsymbol{x}^*,\end{aligned}\right\} \tag{2.2}$$

which allows us to eliminate the pressure field from the dynamic equation for $S_{11,1}$. In particular we find

$$\frac{\partial}{\partial x_1'}\left\langle u_x^2p'\right\rangle=\frac{\rho}{4\pi}\int\frac{\partial}{\partial x_1'}\left(\frac{1}{|\boldsymbol{x}^*-\boldsymbol{x}'|}\right)\frac{\partial^2}{\partial x_m^*\partial x_n^*}\left[\left\langle u_m^*u_n^*u_x^2\right\rangle-\left\langle u_m^*u_n^*\right\rangle\left\langle u_x^2\right\rangle\right]\mathrm{d}\boldsymbol{x}^*$$

and

$$\left\langle u_x\frac{\partial p}{\partial x_1}u_x'\right\rangle=\frac{\rho}{4\pi}\int\frac{\partial}{\partial x_1}\left(\frac{1}{|\boldsymbol{x}^*-\boldsymbol{x}|}\right)\frac{\partial^2}{\partial x_m^*\partial x_n^*}[\langle u_m^*u_n^*u_xu_x'\rangle-\langle u_m^*u_n^*\rangle\left\langle u_xu_x'\right\rangle]\,\mathrm{d}\boldsymbol{x}^*,$$

which yields

$$\frac{\partial S_{11,1}}{\partial t} = -\frac{1}{4\pi}\int \frac{\partial}{\partial x_1'}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}'|}\right)\frac{\partial^2}{\partial x_m^* \partial x_n^*}\left[\langle u_m^* u_n^* u_x^2\rangle - \langle u_m^* u_n^*\rangle\langle u_x^2\rangle\right]\mathrm{d}\boldsymbol{x}^*$$
$$-\frac{1}{2\pi}\int \frac{\partial}{\partial x_1}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}|}\right)\frac{\partial^2}{\partial x_m^* \partial x_n^*}\left[\langle u_m^* u_n^* u_x u_x'\rangle - \langle u_m^* u_n^*\rangle\langle u_x u_x'\rangle\right]\mathrm{d}\boldsymbol{x}^* + \nabla\cdot\langle uuuu\rangle, \tag{2.3}$$

where $\nabla\cdot\langle uuuu\rangle$ represents terms involving the divergence of fourth-order correlations of the type $\langle uuuu'\rangle$ and $\langle uuu'u'\rangle$.

We now follow Batchelor & Proudman (1956) and Davidson (2004) and assume that, at $t = 0$, well-separated points are statistically independent. We then consider what happens immediately after $t = 0$. Here statistical independence is taken to mean that second- and third-order correlations, as well as cumulants of the form

$$[u_i u_j' u_k'' u_l''']_{cum} = \langle u_i u_j' u_k'' u_l'''\rangle - \langle u_i u_j'\rangle\langle u_l''' u_k''\rangle - \langle u_i u_k''\rangle\langle u_l''' u_j'\rangle - \langle u_i u_l'''\rangle\langle u_j' u_k''\rangle,$$

are exponentially small for well-separated points. It follows that, at $t = 0$, the terms on the right of (2.3) of the form $[\langle uuuu\rangle - \langle uu\rangle\langle uu\rangle]$ will tend to zero exponentially fast as $|\boldsymbol{r}^*| = |\boldsymbol{x}^* - \boldsymbol{x}|$ becomes large. Gauss' theorem then allows us to rewrite (2.3) as

$$\frac{\partial S_{11,1}}{\partial t} = -\frac{1}{4\pi}\int \left[\langle u_m^* u_n^* u_x^2\rangle - \langle u_m^* u_n^*\rangle\langle u_x^2\rangle\right]\frac{\partial^2}{\partial x_m^* \partial x_n^*}\frac{\partial}{\partial x_1'}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}'|}\right)\mathrm{d}\boldsymbol{x}^*$$
$$-\frac{1}{2\pi}\int \left[\langle u_m^* u_n^* u_x u_x'\rangle - \langle u_m^* u_n^*\rangle\langle u_x u_x'\rangle\right]\frac{\partial^2}{\partial x_m^* \partial x_n^*}\frac{\partial}{\partial x_1}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}|}\right)\mathrm{d}\boldsymbol{x}^* + \nabla\cdot\langle uuuu\rangle$$

which, because $|\boldsymbol{x}^* - \boldsymbol{x}|$ is symmetric in $\boldsymbol{x}$ and $\boldsymbol{x}^*$, simplifies to

$$\frac{\partial S_{11,1}}{\partial t} = -\frac{1}{4\pi}\int \left[\langle u_m^* u_n^* u_x^2\rangle - \langle u_m^* u_n^*\rangle\langle u_x^2\rangle\right]\frac{\partial^3}{\partial x_1' \partial x_m' \partial x_n'}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}'|}\right)\mathrm{d}\boldsymbol{x}^*$$
$$-\frac{1}{2\pi}\int \left[\langle u_m^* u_n^* u_x u_x'\rangle - \langle u_m^* u_n^*\rangle\langle u_x u_x'\rangle\right]\frac{\partial^3}{\partial x_1 \partial x_m \partial x_n}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}|}\right)\mathrm{d}\boldsymbol{x}^* + \nabla\cdot\langle uuuu\rangle \tag{2.4}$$

Let us now adopt the notation $\boldsymbol{r}^* = \boldsymbol{x}^* - \boldsymbol{x}$ and $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x} = r\hat{\boldsymbol{e}}_x$, while letting r/l become large. Consider the first term on the right of (2.4). It is dominated by contributions in which $\boldsymbol{x}^* \sim \boldsymbol{x}$, since $[\langle u^* u^* uu\rangle - \langle u^* u^*\rangle\langle uu\rangle]$ is small if $\boldsymbol{x}^*$ and $\boldsymbol{x}$ are distant. Thus $|\boldsymbol{r}^*| \ll |\boldsymbol{r}|$ and so we can use the expansion

$$\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}'|} = \frac{1}{|\boldsymbol{r}^* - \boldsymbol{r}|} = \frac{1}{r} - \frac{\partial}{\partial r_i}\left(\frac{1}{r}\right)r_i^* + \frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right)r_i^* r_j^* + O(r^{-4}) \tag{2.5}$$

to evaluate the integral for large r. Note that such an expansion is justified because all integral moments of the term $[\langle uuuu\rangle - \langle uu\rangle\langle uu\rangle]$ are convergent at $t = 0$. To leading order in r, we obtain

$$\frac{\partial S_{11,1}}{\partial t} + \nabla\cdot\langle uuuu\rangle = -\frac{1}{4\pi}\frac{\partial^3}{\partial r_1 \partial r_m \partial r_n}\left(\frac{1}{r}\right)\int \left[\langle u_m^* u_n^* u_x^2\rangle - \langle u_m^* u_n^*\rangle\langle u_x^2\rangle\right]\mathrm{d}\boldsymbol{x}^*$$
$$-\frac{1}{2\pi}\int \left[\langle u_m^* u_n^* u_x u_x'\rangle - \langle u_m^* u_n^*\rangle\langle u_x u_x'\rangle\right]\frac{\partial^3}{\partial x_1 \partial x_m \partial x_n}\left(\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}|}\right)\mathrm{d}\boldsymbol{x}^* + O(r^{-5}).$$

Turning now to the second integral on the right of (2.4), we note that it too is dominated by contributions in which $\boldsymbol{x}^* \sim \boldsymbol{x}$. Moreover, that part of the integrand which takes the form $[\langle uuuu\rangle - \langle uu\rangle\langle uu\rangle]$ can be written as $[u^*_m u^*_n u_x u'_x]_{cum} + \langle u^*_m u'_x\rangle\langle u^*_n u_x\rangle + \langle u^*_m u_x\rangle\langle u^*_n u'_x\rangle$, which is exponentially small for $r \to \infty$. So our expression for $S_{11,1}$ simplifies to

$$\frac{\partial S_{11,1}}{\partial t} + \nabla\cdot\langle uuuu\rangle = -\frac{1}{4\pi}\frac{\partial^3}{\partial r_1 \partial r_m \partial r_n}\left(\frac{1}{r}\right)\int\left[\left\langle u^*_m u^*_n u^2_x\right\rangle - \left\langle u^*_m u^*_n\right\rangle\left\langle u^2_x\right\rangle\right]\mathrm{d}\boldsymbol{x}^*. \quad (2.6)$$

Noting that $\boldsymbol{r} = r\hat{\boldsymbol{e}}_x$, and that $\nabla\cdot\langle uuuu\rangle_\infty$ is exponentially small at $t = 0$, this, in turn, simplifies to

$$\frac{\partial}{\partial t}\left\langle u^2_x u'_x\right\rangle_\infty = \frac{3}{4\pi r^4}\int\left\langle u^2_x\left(2u^2_x - u^2_y - u^2_z\right)^*\right\rangle\mathrm{d}\boldsymbol{x}^*. \quad (2.7)$$

Finally we note that, with the aid of isotropy (and equation (1.2)), (2.7) may be rewritten as

$$\frac{\mathrm{d}^2 I}{\mathrm{d}t^2} = 8\pi\frac{\mathrm{d}}{\mathrm{d}t}[u^3 r^4 K]_\infty = 6\int\langle ss^*\rangle\,\mathrm{d}\boldsymbol{r}' \quad (2.8)$$

where $s = u^2_x - u^2_y$. This is, of course, (1.12), which is the key result of Davidson (2000).

Notice that, so far, our analysis applies only at $t = 0$ when, by virtue of our choice of initial conditions, well-separated points are statistically independent. Let us see if we can extend the analysis to $t > 0$. The key point to note here is that there are two assumptions implicit in (2.7). The first is that fourth-order cumulants at well-separated points fall off faster than r^{-3}, so that the term $\nabla\cdot\langle uuuu\rangle_\infty$ on the left of (2.6) can be ignored. The second assumption is that terms of the form $[\langle uuuu\rangle - \langle uu\rangle\langle uu\rangle]$ in (2.3) decay sufficiently rapidly with $|\boldsymbol{r}^*|$ and $|\boldsymbol{r}|$ for: (i) the use of expansion (2.5) to be legitimate; and (ii) the neglect of the second integral on the right of (2.4) to be valid. Since $[\langle uuuu\rangle - \langle uu\rangle\langle uu\rangle]$ can be rewritten as $[uuuu]_{cum} + 2\langle uu^*\rangle\langle uu^*\rangle$, this requires not only that the fourth-order cumulants decay rapidly, but also that the products of double correlations decay quickly. Both of these requirements are satisfied at $t = 0$ by virtue of our choice of initial conditions, and the question at hand is whether or not they are likely to hold for $t > 0$.

Now it can be shown that the use of expansion (2.5) is legitimate provided that the term $[\langle u^*u^*uu\rangle - \langle u^*u^*\rangle\langle uu\rangle]$ in the first integral on the right of (2.4) decays faster than r^{-7}as $r \to \infty$ (see Appendix A). A similar, though less stringent, condition is also sufficient to ensure that we may neglect of the second integral on the right of (2.4). That is to say, because of the presence of the term $\partial^3|\boldsymbol{x}^* - \boldsymbol{x}|^{-1}/\partial x^3$ in the integrand, and the symmetry of $[\langle u^*u^*uu'\rangle - \langle u^*u^*\rangle\langle uu'\rangle]$ in $\boldsymbol{x}$ and $\boldsymbol{x}'$, this integral is dominated by contributions in which $\boldsymbol{x}^* \sim \boldsymbol{x}$. Thus, for $r \to \infty$, $[\langle u^*u^*uu'\rangle - \langle u^*u^*\rangle\langle uu'\rangle]$ may be approximated by $[\langle uuuu'\rangle - \langle uu\rangle\langle uu'\rangle]$, allowing us to neglect this second integral provided that $[\langle uuuu'\rangle - \langle uu\rangle\langle uu'\rangle]$ decays faster than r^{-4}.

It appears, therefore, that the double correlations do indeed fall off fast enough to justify (2.8). That is, we know that $\langle u^2_x u'_x\rangle_\infty \sim r^{-4}$, and it follows from the Kármán–Howarth equation that the double correlations decay as $\langle u_x u'_x\rangle_\infty \sim r^{-6}$ (see the final sentence of § 1.1.), thus yielding $\langle uu'\rangle_\infty\langle uu'\rangle_\infty \sim r^{-12}$. So the validity, or otherwise, of (2.8) for $t > 0$ is determined simply by the behaviour of the fourth-order cumulants at large separation. If they decay faster than r^{-7}, then (2.8) holds for all t.

It is not at all clear, however, that this restriction on the fourth-order cumulants will, in general, be satisfied for all t. Consider, for example, our neglect of the term $\nabla\cdot\langle uuuu\rangle_\infty$ on the left of (2.6). We dropped this on the assumption that the fourth-order cumulants fall off faster than r^{-4}. While valid at $t = 0$, by virtue of our initial conditions, this assumption seems unlikely to remain valid for arbitrary t, as can be seen from the following argument. Consider the first term on the left of (2.1). Its second derivative with respect to time contains, amongst other things, terms of the form

$$\frac{\partial^2}{\partial t^2}\left\langle \boldsymbol{u}\cdot\nabla\left(u_x^2 u_x'\right)\right\rangle = 2\langle \boldsymbol{u}\cdot\nabla(u_x \dot{u}_x \dot{u}_x')\rangle + \cdots = 2\langle(\boldsymbol{u}\cdot\nabla u_x)\dot{u}_x\dot{u}_x'\rangle$$
$$+ \cdots = 2\langle(\boldsymbol{u}\cdot\nabla u_x)^2 \partial p'/\partial x'\rangle + \cdots$$

where a dot represents a time derivative. Now consider a Taylor expansion of $\langle \boldsymbol{u}\cdot\nabla(u_x^2 u_x')\rangle$ in t about $t=0$. The constant term in the expansion vanishes because of our choice of initial conditions and we have

$$\left\langle \boldsymbol{u}\cdot\nabla\left(u_x^2 u_x'\right)\right\rangle = Ct + [\langle(w^2 \partial p'/\partial x'\rangle + \cdots]_{t=0}\, t^2 + O(t^3), \quad w = \boldsymbol{u}\cdot\nabla u_x$$

for some coefficient C. Now we know from the analysis above that $[\langle u_x^2 \partial p'/\partial x'\rangle + \cdots]_{t=0}$ falls off as r^{-4} at large r. The structural similarity between this term and $[\langle w^2 \partial p'/\partial x'\rangle + \cdots]_{t=0}$ suggests, but does not prove, that the latter term may also fall as r^{-4}, suggesting that $\langle \boldsymbol{u}\cdot\nabla(u_x^2 u_x')\rangle$ will eventually develop an r^{-4} tail. If this is so, the fourth-order cumulants will fall as r^{-4} for $t>0$, which is enough to invalidate (2.7).

In summary then, (2.8) is rigorous at $t=0$ by virtue of our choice of initial conditions. It also applies for $t>0$ if, but only if, the fourth-order cumulants decay faster than r^{-7} for well-separated points. As noted in § 1.3, there is some experimental evidence to suggest that, in fully developed turbulence, the cumulants are indeed very small for large separation, though it is unclear whether or not they fall off faster than r^{-7}. On the other hand, the argument above tentatively suggests that the cumulants could, under certain conditions, fall as r^{-4}. If this is so, then (2.8) cannot be applied to turbulence emerging from arbitrary initial conditions. We shall see that our DNS suggests that (2.8) is, in general, invalid.

3. The numerical evidence

The direct numerical simulations reported here employ the spectral code described in Kaneda *et al* (2004). The boundary conditions are periodic and the random initial incompressible flow was chosen from a Gaussian ensemble with $\langle \boldsymbol{u}^2\rangle_{t=0}=1$ and the prescribed isotropic energy spectrum $E \sim k^4 \exp[-2(k/k_p)^2]$. The box size is 2π in each direction and so the lowest wavenumber is $k_{min}=1$. The details of the simulations are given in table 1. Here N is the number of grid points in each direction of the periodic box, $k_p$is the wavenumber at which $E(k,t=0)$ peaks, time is normalized by the initial eddy turnover time, $1/\langle \boldsymbol{u}^2\rangle_0^{1/2} k_p = 1/k_p$, the initial Reynolds number Re is based on the integral scale $l=k_p^{-1}$, so that $Re=1/(k_p\nu)$, and the simulations were continued up to the time $t_{max.}$ Note that, with this choice of integral scale, the initial value of l_{box}/l is $2\pi k_p$. (An alternative definition of the integral scale, that of the integral of the longitudinal correlation function, would have l a factor of $(2\pi)^{1/2}$ larger.) Since l grows approximately as $t^{2/7}$, and $t_{max}=300$, we would expect l to grow by a factor of ~ 5 during the simulations. Thus, by the end of the simulations we would expect $l_{box}/l \sim 2\pi k_p/5 \sim k_p$.

The so-called phase-shift method has been used for de-aliasing, in which the maximum wavenumber k_{max} of the retained Fourier modes is about $2^{1/2}N/3$, leading to the approximate ratio of k_{max}/k_p given in table 1. The spectrum $E(k)$ was calculated from shell averages, as described in Appendix B. Finally we note that

Run number	N	k_p	Re	t_{max}	k_{max}/k_p
1	1024	80	62.5	300	6
2	1024	40	250	300	12
3	1024	40	125	300	12
4	1024	80	31.3	300	6
5	1024	40	62.5	300	12
6	512	40	31.3	300	6
7	256	20	125	300	6
8	256	20	62.5	300	6

TABLE 1. Details of the simulations.

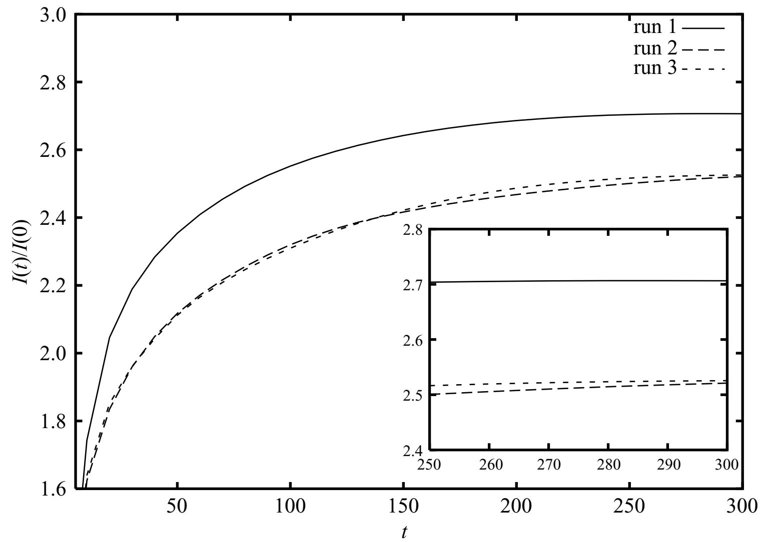

FIGURE 2. $I(t)/I(0)$ versus time for run 1 ($k_p = 80$, $Re = 62.5$), run 2 ($k_p = 40$, $Re = 250$) and run 3 ($k_p = 40$, $Re = 125$).

the values of $I(t)$ and d^2I/dt^2 were estimated by fitting the curves $E = Ik^4/24\pi^2$ and $\partial^2E/\partial t^2 = (d^2I/dt^2)k^4/24\pi^2$ to the data in the vicinity of $k = 0.08k_p$.

Figures 2, 3, 4 and 5 show the results of run 1 ($k_p = 80$, $Re = 62.5$), run 2 ($k_p = 40$, $Re = 250$) and run 3 ($k_p = 40$, $Re = 125$). Figure 2 shows $I(t)/I(0)$ for the three runs, and figure 3 the corresponding exponents, $m(t)$, in the decay law $u^2 \sim t^{-m}$. Figures 4 and 5 show the evolution of $E(k, t)$ and d^2I/dt^2 for run 1.

We observe, in figure 4, that $E(k, t)$ maintains its $E \sim k^4$ form, as expected. Moreover, figure 2 shows that, after an initial transient, $I(t)/I(0)$ settles down to an (almost) constant value in runs 1 and 3. This is achieved by $t \sim 250$, and so we might refer to $t > 250$ as the mature, or fully developed, state. (While there is still some time dependence in run 2 at $t = 300$, this is small.) Figure 5 shows that, for $30 < t < 200$, d^2I/dt^2 scales approximately as t^{-2}, but falls faster than t^{-2} as we approach the fully developed state. This implies that, during the transition period, $30 < t < 200$, I varies as $I(t) \sim \ln t$. Thus the transient approach to the asymptotic state is approximately logarithmic. Moreover, the fact that $d^2I/dt^2 < 0$ for $t > 10$ confirms that, during this

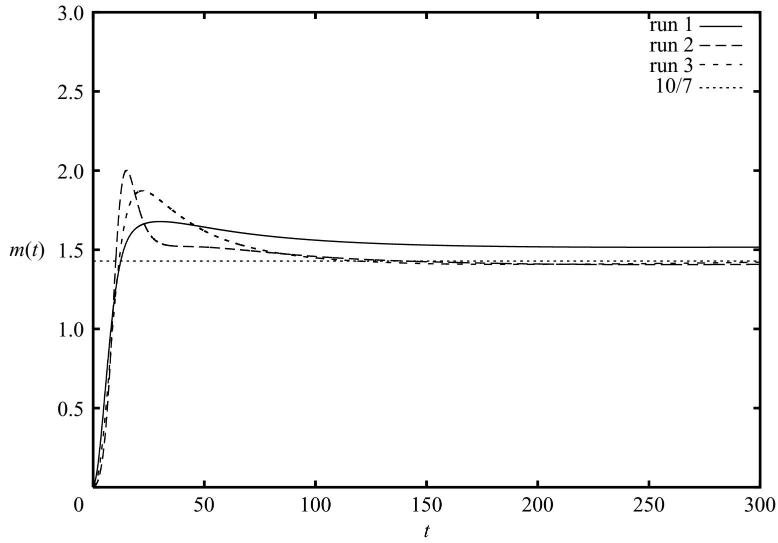

FIGURE 3. $m(t)$ versus time for run 1 ($k_p = 80$, $Re = 62.5$), run 2 ($k_p = 40$, $Re = 250$) and run 3 ($k_p = 40$, $Re = 125$).

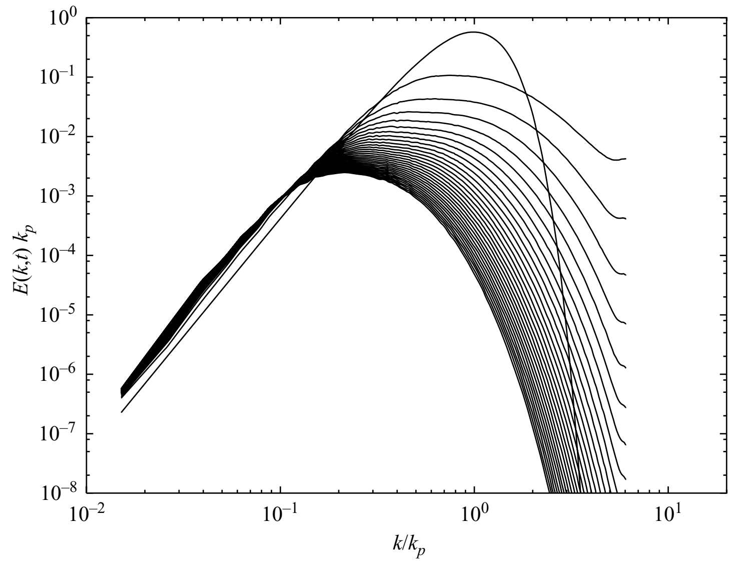

FIGURE 4. $E(k, t)$ for run 1 ($k_p = 80$, $Re = 62.5$) from $t = 0$ up to $t = 300$ in steps of $\Delta t = 10$.

period, (2.8) is not valid, the implication being that the fourth-order cumulants do not fall off fast enough for (2.8) to hold. (We shall return to these points in § 4.)

As $I(t)$ settles down to an (almost) constant value, so does $m(t)$. In the case of runs 2 and 3, which have the highest values of Re, m approaches the Kolmogorov law, $m = 10/7$. For run 1, however, m asymptotes to a slightly higher value of ~ 1.5. In view of (1.18), which tells us that any time dependence of $I(t)$ should reduce m below $10/7$, we might anticipate that the behaviour of m in run 1 is a consequence of the low value of Re. We shall confirm this next.

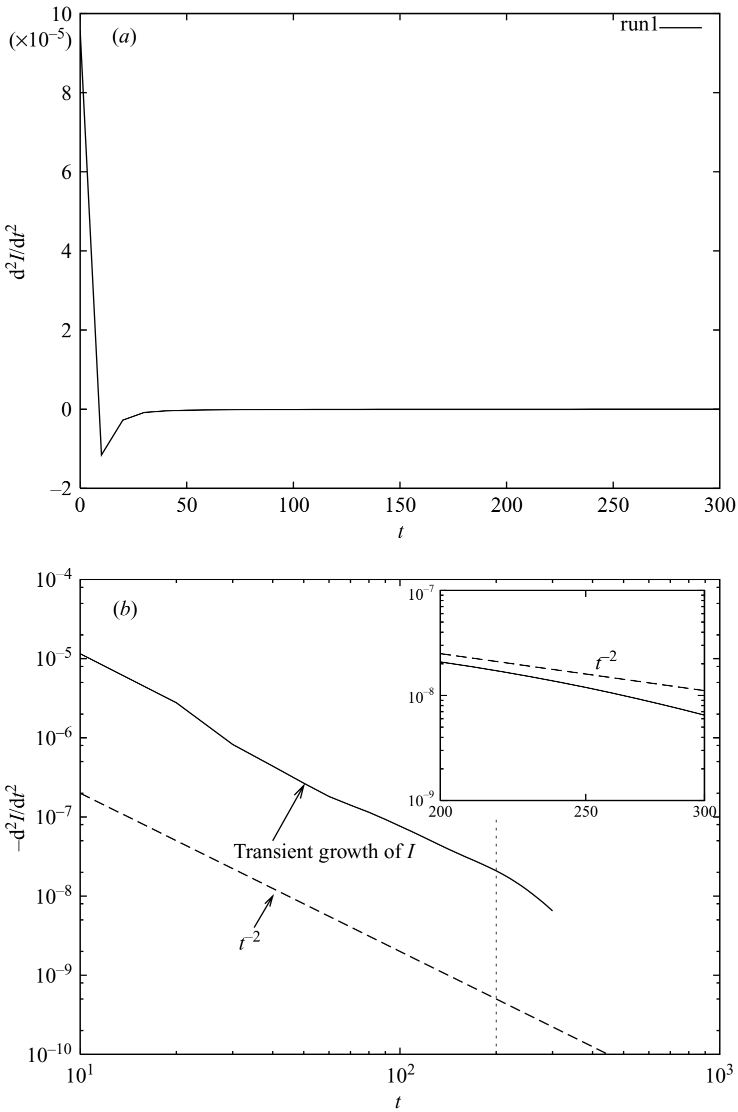

FIGURE 5. $\mathrm{d}^2I/\mathrm{d}t^2$ for run 1 ($k_p = 80$, $Re = 62.5$) from $t = 0$ up to $t = 300$. (*a*) Lin-lin plot of $\mathrm{d}^2I/\mathrm{d}t^2$. (*b*) Log-log plot of $-\mathrm{d}^2I/\mathrm{d}t^2$. Note that $\mathrm{d}^2I/\mathrm{d}t^2 \sim -t^{-2}$ during the transient phase $30 < t < 200$, but that $\mathrm{d}^2I/\mathrm{d}t^2$ falls faster than t^{-2} for $t > 250$.

In order to test the hypothesis that the deviation of $m(t)$ from 10/7 in run 1 is a viscous effect, we have plotted $m(t)$ for runs 1 to 6, in which Re varies from 31.3 up to 250 (figure 6). We observe that, for the runs at $Re = 31.3$ (runs 4 and 6) $m \to \sim 1.63$, for $Re = 62.5$ (runs 1 and 5) $m \to \sim 1.50$, while $m \to \sim 10/7$ for $Re = 125$ and $Re = 250$. This suggests that Re must exceed ~ 100 in order to observe the 10/7 decay law, irrespective of the behaviour of $I(t)$.

Let us now consider the influence of k_p on the results. Figure 7 shows $E(k,t)$ for $Re = 62.5$ and $k_p = 20,\ 40,\ 80$ (runs 1, 5 and 8). We observe that the evolution of the small to intermediate scales is more or less independent of k_p, suggesting that, if $k_p \geqslant 20$, periodicity is not a problematic boundary condition for those interested in

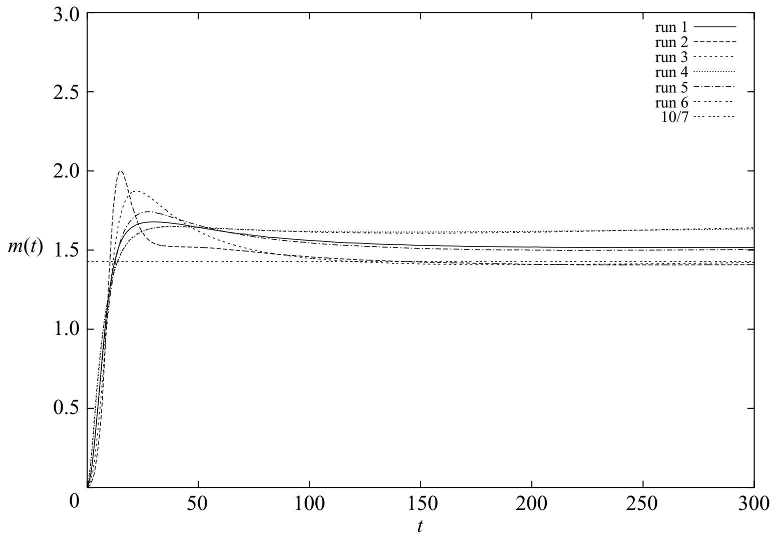

FIGURE 6. $m(t)$ versus time for runs 1 to 6. $Re = 31.3$ in runs 4 and 6, $Re = 62.5$ in runs 1 and 5, $Re = 125$ in run 3, and $Re = 250$ in run 2.

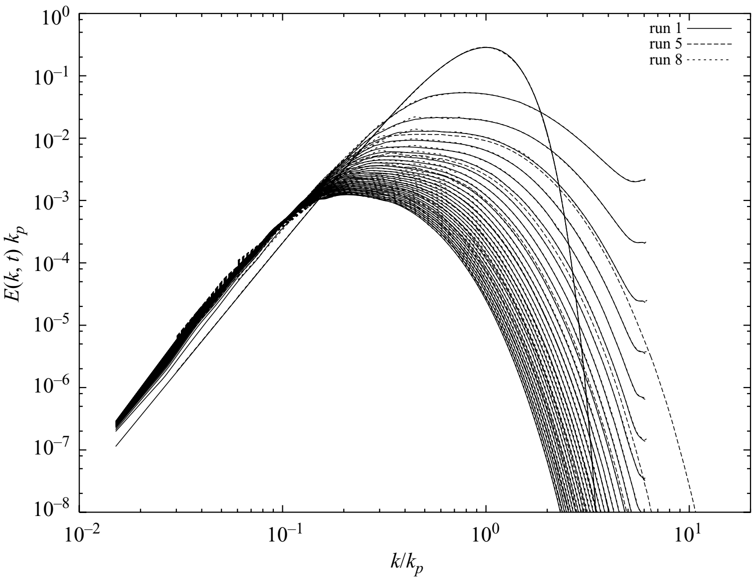

FIGURE 7. $E(k, t)$ for runs 1 and 8 ($Re = 62.5$, $k_p = 20$, 80) for $t = 0$ up to $t = 300$, in steps of $\Delta t = 10$. $E(k, t)$ for run 5 ($Re = 62.5$, $k_p = 40$) for $t = 0$ up to $t = 288$, in steps of $\Delta t = 32$.

the equilibrium range. Whether or not periodicity influences the evolution of $I(t)$ for small k_p is hard to say, since there is no discernible $E \sim k^4$ region for $k_p \leqslant 20$. This is clear from figure 8 which shows $E(k)$ at $t = 300$ for the same three runs. Figure 9 shows $E(k, t)$ for $Re = 125$ and $k_p = 20$, 40 (runs 3 and 7). Again we observe that the small to intermediate scales are independent of k_p, but that there is no discernible $E \sim k^4$ region for $k_p \leqslant 20$.

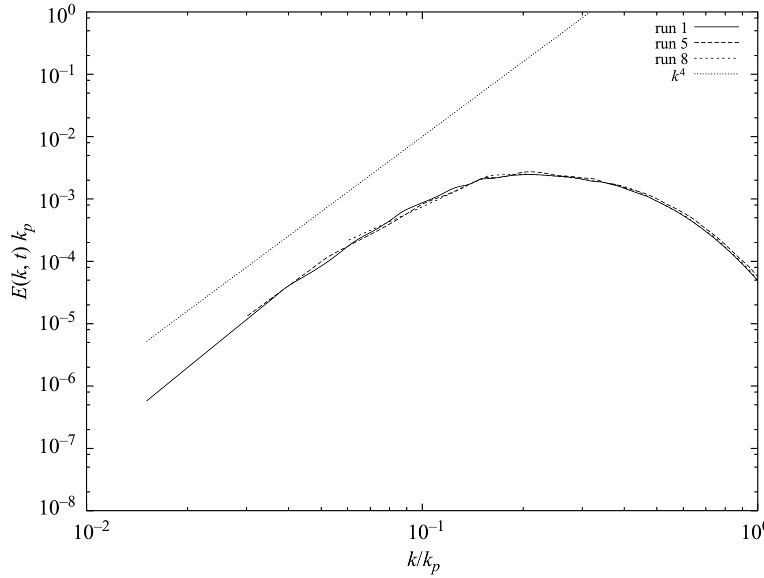

FIGURE 8. $E(k)$ at $t = 300$ for runs 1 and 8 ($Re = 62.5$, $k_p = 20$, 80). $E(k)$ at $t = 288$ for run 5 ($Re = 62.5$, $k_p = 40$).

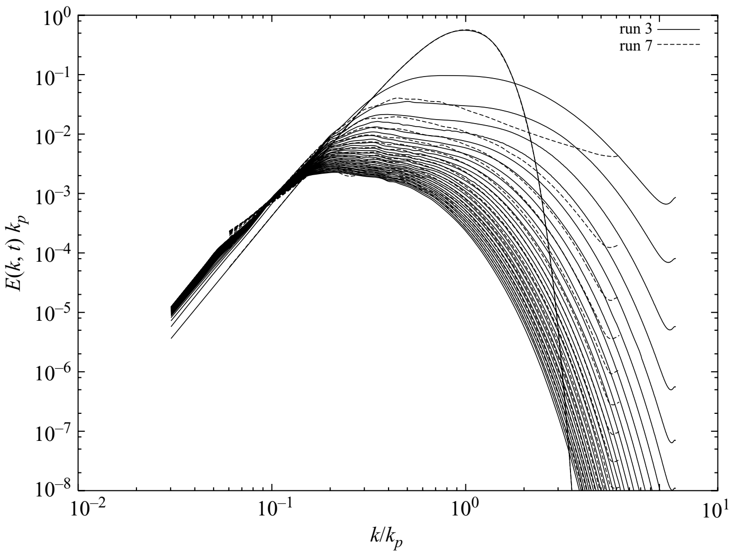

FIGURE 9. $E(k, t)$ for run 3 ($Re = 125$, $k_p = 40$) from $t = 0$ up to $t = 300$, in steps of $\Delta t = 10$. $E(k, t)$ for run 7 ($Re = 125$, $k_p = 20$) from $t = 0$ up to $t = 300$, $\Delta t = 20$.

Finally, figure 10 shows the normalized two-point vorticity correlation $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle (r, t)/\langle \boldsymbol{\omega}^2 \rangle$, at different times in run 1 ($Re = 62.5$, $k_p = 80$). Let us consider the curve corresponding to $t = 300$. The behaviour of $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle / \langle \boldsymbol{\omega}^2 \rangle$ for $r \lesssim 0.2$ (i.e. $k_p r \lesssim 16$) is approximately of the form $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle / \langle \boldsymbol{\omega}^2 \rangle \sim \exp[-a(k_p r)^2]$. The correlation goes negative at $r \sim 0.23$ ($k_p r \sim 18$) and then slowly decays in magnitude. The maximum (negative) value of the correlation in the range $k_p r > 18$ is 0.01, falling to 0.001 for

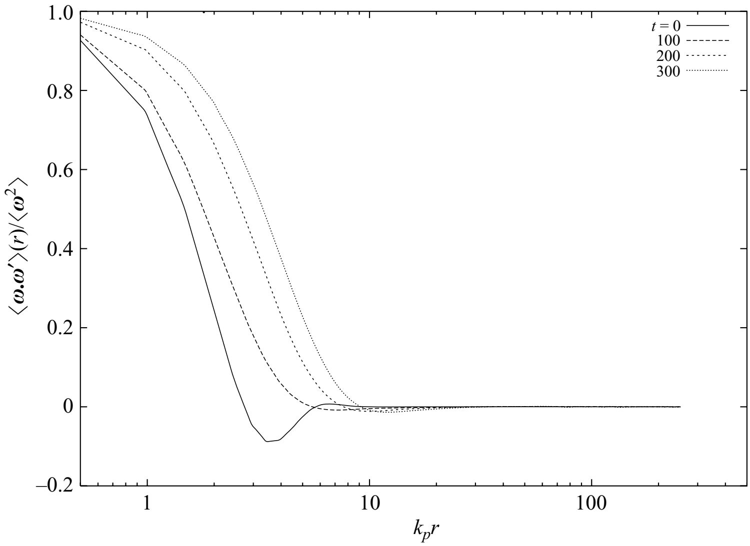

FIGURE 10. $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle(r)/\langle \boldsymbol{\omega}^2 \rangle$ versus r at $t = 0$, 100, 200, 300 for run 1 ($Re = 62.5$, $k_p = 80$).

$k_p r > 65$. Because the ratio of noise to signal becomes large for $k_p r \gg 1$, it is difficult to detect the power-law behaviour $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle(r, t)/\langle \boldsymbol{\omega}^2 \rangle \sim A(t)(k_p r)^{-8}$ predicted by Batchelor & Proudman (1956) for $k_p r \gg 1$. However plots of $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$, obtained by inverting its Fourier transform partner, show some evidence that $\langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle \sim r^{-6}$, which is consistent with $\langle \boldsymbol{\omega} \cdot \boldsymbol{\omega}' \rangle \sim r^{-8}$.

4. Discussion and conclusions

We have found that I tends to an (almost) constant value as the turbulence matures, which indicates that the long-range interactions, as measured by $[r^4 K]_\infty$, die away. Thus, at large times, we approach the classical (pre-Batchelor & Proudman) picture of fully developed homogenous turbulence. It seems likely that this partial suppression of the long-range triple correlations is a direct consequence of the gradual change in the morphology of the vorticity field. Initially, when we have random Gaussian modes, the vorticity field is devoid of structure. At later times the Navier–Stokes equation has had a chance to pull the vorticity field into some kind of asymptotic state, the precise form of which will depend on Re.

Quite why the long-range triple correlations are so weak in fully developed turbulence we cannot say. We merely make the following observations. The Kármán–Howarth equation (1.1) can be written in the form

$$\frac{\partial}{\partial t}\langle u_x u_x' \rangle = \frac{1}{2}\left\langle \frac{\partial u_x}{\partial t} u_x' \right\rangle = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \langle u_x^2 u_x' \rangle + \nu(\sim) \tag{5.1}$$

from which

$$2\frac{\partial^2}{\partial t^2}\langle u_x u_x' \rangle = \left\langle \frac{\partial u_x}{\partial t}\frac{\partial u_x'}{\partial t} \right\rangle + \left\langle \frac{\partial^2 u_x}{\partial t^2} u_x' \right\rangle = \frac{1}{r^4}\frac{\partial}{\partial r} r^4 \left[\left\langle u_x^2 \frac{\partial u_x'}{\partial t} \right\rangle + \left\langle 2u_x \frac{\partial u_x}{\partial t} u_x' \right\rangle \right] + \nu(\sim). \tag{5.2}$$

The two inviscid terms on the right of (5.2) lead directly to the two integrals on the right of (2.3). In the analysis of Batchelor & Proudman (1956), as well as the quasi-normal closure model, the contribution from the term $\langle u_x' \partial^2 u_x/\partial t^2 \rangle$ is zero, corresponding to the neglect of the second integral on the right of (2.4). In our DNS, however, we find that the contribution from $\langle u_x' \partial^2 u_x/\partial t^2 \rangle$ is non-zero, and that the decline of $[r^4 K]_\infty$corresponds not so much to the vanishing of the contributions from $\langle u_x' \partial^2 u_x/\partial t^2 \rangle$ and $\langle (\partial u_x/\partial t)(\partial u_x'/\partial t) \rangle$ individually, but rather to the cancellation of these two terms. Thus the fact that, at large times, $[r^4 K]_\infty$ is small need not imply that long-range pressure–velocity correlations of the form $\langle u_x^2 p' \rangle_\infty$vanish.

We might also note that the approximate t^{-2} decline of $\mathrm{d}^2 I/\mathrm{d}t^2$ at intermediate times $(30 < t < 200)$ could be interpreted as the large scales evolving in an approximately self-similar manner. That is, the combination of (1.2) and (2.3) tells us that, if the large scales are self-similar,

$$\mathrm{d}^2 I/\mathrm{d}t^2 \sim -u^4 l^3 \sim -I u^2/l^2 \sim -I/t^2$$

since the integral scale, l, scales as $l \sim ut$. At intermediate times I varies rather slowly, confined to the range 2.2–2.7 in run 1. So self-similarity predicts something close to $\mathrm{d}^2 I/\mathrm{d}t^2 \sim -1/t^2$ during the transient. However this interpretation is not entirely satisfactory since the flow structure is still evolving during this period and so we have no right to assume self-similarity.

We end by noting that the question of the conservation (or lack of conservation) of $I(t)$ in mature turbulence is not just relevant to isotropic turbulence. Many other systems, such as homogeneous MHD turbulence, and rotating stratified turbulence, conserve one or more components of angular momentum. The fact that there is evidence of the suppression of long-range velocity correlations in isotropic turbulence gives us hope that these long-range interactions are also weak in these more complex flows. If so, they will possess Loitsyansky-like invariants of the form of (1.4). (See the discussion in Davidson 1997, 2004.)

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Appendix A. The legitimacy of using expansion (2.5)

The key assumption inherent in (2.6) is that we can use the expansion

$$\frac{1}{|\boldsymbol{x}^* - \boldsymbol{x}'|} = \frac{1}{|\boldsymbol{r}^* - \boldsymbol{r}|} = \frac{1}{r} - \frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) r_i^* + \frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right) r_i^* r_j^* + O(r^{-4}) \qquad \text{(A 1)}$$

to evaluate the leading-order far-field term of the non-local pressure equation

$$\langle u_p(0) u_q(0) p(\boldsymbol{x}') \rangle = \frac{\rho}{4\pi} \int \frac{\partial^2}{\partial x_m^* \partial x_n^*} [\langle u_p(0) u_q(0) u_m^* u_n^* \rangle - \langle u_p(0) u_q(0) \rangle \langle u_m^* u_n^* \rangle] \frac{\mathrm{d}\boldsymbol{x}^*}{|\boldsymbol{x}^* - \boldsymbol{x}'|}.$$

(Here, for simplicity, we take $\boldsymbol{x} = 0$ so that $\boldsymbol{r}^* = \boldsymbol{x}^* - \boldsymbol{x} = \boldsymbol{x}^*$ and $\boldsymbol{r} = \boldsymbol{x}' - \boldsymbol{x} = \boldsymbol{x}'$.) In this appendix we seek to establish the conditions under which this procedure is legitimate.

Let us start by rewriting the integral equation as

$$I_{pq}(\boldsymbol{x}') = \langle u_p(0)u_q(0)p(\boldsymbol{x}')\rangle = \int \frac{\partial^2}{\partial x_m^* \partial x_n^*}[U_{mnpq}(\boldsymbol{x}^*)]\frac{\mathrm{d}\boldsymbol{x}^*}{|\boldsymbol{x}^* - \boldsymbol{x}'|}. \tag{A 2}$$

From now on we shall drop the subscripts p and q, leaving them as understood. Clearly, the use of expansion (A 1) is legitimate if $U_{mn}(\boldsymbol{x}^*)$ decreases rapidly with $r^* = |\boldsymbol{x}^*|$. Let us suppose that $U_{mn}(\boldsymbol{x}^*) \sim (r^*)^{2-n}$ for $r^* \gg l$, so that the integrand in (A 2) decays as $(r^*)^{-n}$. We shall now show that the use of (2.5) is legitimate provided that $n > 9$.

Let us take $r = |\boldsymbol{x}'| \ggg l$ and divide the domain of integration into three regions: V_1 is an inner sphere of radius a, where $l \ll a \ll r$; V_2 is a spherical annulus of inner radius a and outer radius $r/2$; and V_3 is the rest of space. Next, for ease of notation, we normalize all length scales by dividing by the integral scale l. In normalized units we have $r \ggg 1$ and $1 \ll a \ll r$. We now choose $a = r^\alpha$, $\alpha < 1$.

Let us consider the contribution to I from region V_1, say I_1. Since the domain is bounded we may apply (A 1). Moreover, the fourth-order term in (A 1) is no greater than $\sim a^3/r^4$, and so we find,

$$I_1(\boldsymbol{x}') = \frac{1}{r}\int_{V_1} \frac{\partial^2 U_{mn}}{\partial x_m^* \partial x_n^*}\,\mathrm{d}\boldsymbol{x}^* - \frac{\partial}{\partial r_i}\left(\frac{1}{r}\right)\int_{V_1} x_i^* \frac{\partial^2 U_{mn}}{\partial x_m^* \partial x_n^*}\,\mathrm{d}\boldsymbol{x}^*$$
$$+\frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right)\int_{V_1} x_i^* x_j^* \frac{\partial^2 U_{mn}}{\partial x_m^* \partial x_n^*}\,\mathrm{d}\boldsymbol{x}^* + O(a^3/r^4). \tag{A 3}$$

Let us rewrite this as

$$I_1 = G + O(a^3/r^4) \tag{A 4}$$

where G represents the three integrals on the right of (A 3). Notice that the first two integrals contained in G can be rewritten as surface integrals, which are of the order of a^{3-n}/r and a^{4-n}/r^2 respectively.

The contributions from V_2 and V_3 are readily estimated. Since the integrand in (A 2) falls off as $(r^*)^{-n}$ in V_2 and V_3 we find that the corresponding integrals are, at most, of the order of $I_2 \sim a^{3-n}/r$ and $I_3 \sim r^{2-n}$, and so

$$r^3|I - G| < C_1\frac{a^3}{r} + C_2\frac{r^2}{a^{n-3}} = \frac{C_1}{r^{1-3\alpha}} + \frac{C_2}{r^{\alpha(n-3)-2}} \tag{A 5}$$

where C_1 is a constant, which could be zero. We now choose $\alpha = 3/n$. The two remainder terms on the right of (A 5) are then of equal order and tend to zero as $r \to \infty$, provided that $n > 9$. Thus, for $n > 9$, $I - G$ is smaller than $O(r^{-3})$. Under the same conditions the first two integrals on the right of (A 3), which can be rewritten as surface integrals, make contributions to (A 5) of the same order as (or smaller than) the remainder terms. We conclude that, provided $n > 9$,

$$I(\boldsymbol{x}') - \frac{1}{2}\frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right)\int_{V_1} x_i^* x_j^* \frac{\partial^2 U_{mn}}{\partial x_m^* \partial x_n^*}\,\mathrm{d}\boldsymbol{x}^*$$

is also smaller than $O(r^{-3})$. Finally we rewrite the integrand of the integral above as the sum of a divergence plus $2U_{ij}$. Once again we find that the surface integral makes a contribution to (A 5) which is no larger than the remainder terms and so

$$I(\boldsymbol{x}') - \frac{\partial^2}{\partial r_i \partial r_j}\left(\frac{1}{r}\right)\int_{V_1} U_{ij}\,\mathrm{d}\boldsymbol{x}^*$$

is also smaller than $O(r^{-3})$. This is sufficient to justify the use of expansion (2.5). We conclude therefore, that (2.8) is justified provided that the fourth-order cumulants decay faster than r^{-7}.

Appendix B. The method of shell averaging

In order to compute the band-averaged energy spectrum we first computed the average $e(q)$ over the surface of the sphere of radius q ($q^2 = 1, 2, 3 \ldots\ldots$) in wave-vector space,

$$e(q) = \frac{1}{N_q} \sum_{|\boldsymbol{k}|=q} \hat{\boldsymbol{u}}(\boldsymbol{k}) \cdot \hat{\boldsymbol{u}}(-\boldsymbol{k}),$$

where $\hat{\boldsymbol{u}}(\boldsymbol{k})$ is the Fourier transform of the velocity field, N_p is the number of grid points on the shell, and $\sum_{|\boldsymbol{k}|=p}$ denotes the summation over the grid points on the shell. We then compute an approximation for the band average spectrum E at k' by

$$\frac{E(k')}{2\pi k'^2} \approx \frac{1}{n_{shell}} \sum_{band}^{k} e(q),$$

where n_{shell} is the number of shells in the range $k - 1/2 \leqslant q < k + 1/2$, $\sum_{band}^{k}$ denotes the sum over the range, i.e. $\sum_{band}^{k} \equiv \sum_{k-1/2 \leqslant q < k+1/2}$, with $k = 1, 2, 3 \ldots\ldots$, and $k' \equiv \frac{1}{n_{shell}} \sum_{band}^{k} q$.

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